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Bidirectional Quantum Teleportation of an Arbitrary Number of Qubits by Using Four Qubit Cluster State

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Abstract
This paper presents a new bidirectional quantum teleportation (BQT) protocol in which users can simultaneously transmit an arbitrary number of particles to each other via a four-qubit cluster state as quantum channel. With the use of auxiliary qubits and controlled-not gates, transformation circuits are introduced which transform n-qubit states into single superposition state and then teleport them over the channel by using Bell-state measurement and applying a proper Pauli operator. At last, with the aid of another transformation circuit, the initial states can be reconstructed. With this method, the BQT purpose is realized more advantageous compared to recent presented protocols.

Keywords Bidirectional quantum teleportation · Four-qubit cluster state · Bell-state measurement · Single superposition state · State-reduction circuit · State-reconstruction circuit

1 Introduction

Quantum Teleportation (QT) is a branch of quantum information theory which allows the transmission of unknown quantum states over a spatial distance using a previously shared entanglement and classical communication. In 1993, Bennett et al. proposed the first teleportation protocol between Alice and Bob [1]. Later in 1997, the first experimental demonstration of quantum teleportation using entangled photons was proposed by Bouwmeester based on Bennett idea [2]. After that, different QT protocols were proposed using Greenberger-Horne-Zeilinger (GHZ) state, Einstein-Podolsky-Rosen (EPR) pair and other entangled states as a quantum channel [3–7].

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Over the past years, different types of teleportation schemes were proposed such as: quantum controlled teleportation (QCT), bidirectional quantum teleportation (BQT) and bidirectional quantum controlled teleportation (BQCT). In BQT protocol, Alice and Bob both can play the role of sender and receiver and communicate an unknown quantum state to each other. First realization of BQT protocol via five-qubit cluster state was presented by Zha et al. in 2013 [8]. Majority of schemes use BQT or BCQT with establishment of a variety of teleportation procedures, such as increasing number of transmitted qubit states or reducing number of qubits in quantum channel [9–16], multi controlling communication for increasing security of the connection [17] and asymmetric QCT, which number of qubit states in sender and receiver sites are not the same [18–23]. Other schemes use hybrid communications which QCT emerging with remote state preparation (RSP) protocol and prepare a state for one of the parties [24–27]. Entanglement swapping was done in [28]. In [29–31] cyclic and circular QT and QCT are investigated in which Alice teleports a qubit to Bob, Bob teleports a qubit to Charlie and Charlie teleports a qubit back to Alice. In [32] study, any of the three participants can play the supervisor role. Some recent developments presented general protocols for teleportation of n-arbitrary qubit states [33, 34]. Zhou et al. published a few schemes in which 3-qubit entangled states can be transformed into 2-qubit entangled states and single superposition qubit state using Toffoli and CNOT operators and then reconstruct the initial qubits at the receiver site [35–37]. In 2018, Xiao et al. presented bidirectional controlled quantum teleportation in the three-dimension system which utilize qutrits and perform generalized Bell-state measurements [38].

In previous schemes, users were able to teleport a single qubit state using a quantum channel with two-qubit state [39]. As the transmissible qubits increased, the channel width gradually expanded. In this paper, as an extension of the study in [36], we implement a scheme for bidirectional quantum teleportation of an arbitrary number of qubits in which users are able to teleport an arbitrary number of particles (n) to each other using state-transformation circuits and a four-qubit cluster state. By carrying out the following steps, the BQT task is accomplished: Firstly, n-qubit states are transformed into a single superposition state, using a state-transformation circuit and auxiliary qubits. Secondly, with the aid of Bell-state measurement (BSM) and applying proper unitary transformation, single superposition states are teleported using four-qubit quantum channel. Lastly, another state-reconstruction circuit with the aid of auxiliary qubits is used to transform the communicated single superposition states into the initial arbitrary number of states.

The coming up sections are organized as follows: Sect. 2 describes the proposed BQT protocol, Sect. 3 compares the efficiency of the scheme to recent works and Sect. 4 concludes the paper and discusses future works.

2 BQT of an Arbitrary Number of Qubits

In this paper, Alice and Bob transmit an arbitrary number of qubits in a BQT protocol using four-qubit cluster state. At first, quantum channel and transmissible target qubits must be prepared for BQT to be realized. Suppose that Alice and Bob states of qubits are represented in Eqs. (1) and (2) respectively.

$$|\Phi\rangle_{A_1A_2\ldots A_n} = \sum_{i=1}^{2^n-1} \alpha_i |i\rangle \tag{1}$$

$$|\Phi\rangle_{B_1B_2\ldots B_n} = \sum_{j=1}^{2^n-1} \beta_j |j\rangle \tag{2}$$

$$\sum_{i=1}^{n} |\alpha_i|^2 = \sum_{j=1}^{n} |\beta_j|^2 = 1 \tag{3}$$

Where α_i and β_j are arbitrary unknown coefficients that satisfy Eq. (3). Alice holds qubits $A_1A_2\ldots A_n$ and Bob owns $B_1B_2\ldots B_n$ particles. The quantum channel for establishing the bidirectional teleportation which is a four-qubit cluster state shared by Alice and Bob, is expressed in Eq. (4).

$$|\Phi\rangle_{a_1a_2b_1b_2} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{a_1a_2b_1b_2} \tag{4}$$

The initial joint state of the senders and quantum channel is expressed as

$$\Psi_{A_1A_2\ldots A_nB_1B_2\ldots B_na_1a_2b_1b_2} = |\Phi\rangle_{A_1A_2\ldots A_n} \otimes |\Phi\rangle_{B_1B_2\ldots B_n} \otimes |\Phi\rangle_{a_1a_2b_1b_2} \tag{5}$$

Since the width of quantum channel is already determined, Alice and Bob can maximally transmit single-qubit states simultaneously, therefore in order to transmit arbitrary number of n-qubits in the same channel, Alice and Bob's n-qubit states need to be transformed into the tensor product of $(n-1)$ qubit entangled states and an initialized 0-qubit state. The modality can be accomplished if we transform $(n-1)$ remaining qubit states into the tensor product of $(n-2)$ qubit entangled states and an initialized 0-qubit state. Repeating this procedure, final transformation will lead into the tensor product of single superposition state and a 0-qubit state.

Figure 1 shows the required transformation circuit Alice and Bob can use to transform their n-qubit states into one single superposition state using $(n-1)$ auxiliary 0-qubit states. Mathematically, this state transformation can be shown for Alice using expressed relations in Eq. (6) and let $|\Phi\rangle_{A_1}$ and $|\Phi\rangle_{B_1}$ to be transmissible superposition qubit sates in the quantum teleportation protocol which they contain the whole information of arbitrary n-qubit states. The coefficients α_0, α_1, β_0, β_1 used in Eq. (7) are unknown but $|\alpha_0|^2 + |\alpha_1|^2 = 1$ and $|\beta_0|^2 + |\beta_1|^2 = 1$.

$$\begin{array}{lll}
|\Phi\rangle_{A_1A_2\ldots A_n} & (n-1)\,Controlled\text{–}Not \xrightarrow{} & |\Phi\rangle_{A_1A_2\ldots A_{n-1}} \otimes |0\rangle_{A_n} \\
|\Phi\rangle_{A_1A_2\ldots A_{n-1}} & (n-2)\,Controlled\text{–}Not \xrightarrow{} & |\Phi\rangle_{A_1A_2\ldots A_{n-2}} \otimes |0\rangle_{A_{n-1}} \\
 & \cdots & \\
 & \cdots & \\
 & \cdots & \\
|\Phi\rangle_{A_1A_2A_3} & Toffoli \xrightarrow{} & |\Phi\rangle_{A_1A_2} \otimes |0\rangle_{A_3} \\
|\Phi\rangle_{A_1A_2} & CNOT \xrightarrow{} & |\Phi\rangle_{A_1} \otimes |0\rangle_{A_2}
\end{array} \tag{6}$$

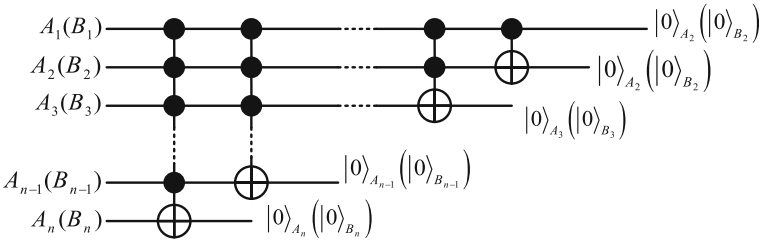

Fig. 1 Quantum circuit for transformation of a class of n-qubit states into a single superposition state using auxiliary qubits and controlled-not gates

$$\begin{aligned} |\Phi\rangle_{A_1} &= \left(\alpha_0|0\rangle_{A_1} + \alpha_1|1\rangle_{A_1}\right) \\ |\Phi\rangle_{B_1} &= \left(\beta_0|0\rangle_{B_1} + \beta_1|1\rangle_{B_1}\right) \end{aligned} \tag{7}$$

Final state of the whole system can be expressed as Eq. (8) where Bell-states are defined as $\Phi^{\pm} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $\Psi^{\pm} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

$$\begin{aligned} &\Psi_{A_1B_1a_1a_2b_1b_2} = |\Phi\rangle_{A_1} \otimes |\Phi\rangle_{B_1} \otimes |\Phi\rangle_{a_1a_2b_1b_2} \\ &= \frac{1}{\sqrt{4}}\Big[|\Psi^+\rangle_{A_1a_1}|\Psi^+\rangle_{B_1b_2}\big|\chi^1\big\rangle_{a_2b_1} + |\Psi^+\rangle_{A_1a_1}|\Psi^-\rangle_{B_1b_2}\big|\chi^2\big\rangle_{a_2b_1} \\ &\quad +|\Psi^+\rangle_{A_1a_1}|\Phi^+\rangle_{B_1b_2}\big|\chi^3\big\rangle_{a_2b_1} + |\Psi^+\rangle_{A_1a_1}|\Phi^-\rangle_{B_1b_2}\big|\chi^4\big\rangle_{a_2b_1} \\ &\quad +|\Psi^-\rangle_{A_1a_1}|\Psi^+\rangle_{B_1b_2}\big|\chi^5\big\rangle_{a_2b_1} + |\Psi^-\rangle_{A_1a_1}|\Psi^-\rangle_{B_1b_2}\big|\chi^6\big\rangle_{a_2b_1} \\ &\quad +|\Psi^-\rangle_{A_1a_1}|\Phi^+\rangle_{B_1b_2}\big|\chi^7\big\rangle_{a_2b_1} + |\Psi^-\rangle_{A_1a_1}|\Phi^-\rangle_{B_1b_2}\big|\chi^8\big\rangle_{a_2b_1} \\ &\quad +|\Phi^+\rangle_{A_1a_1}|\Psi^+\rangle_{B_1b_2}\big|\chi^9\big\rangle_{a_2b_1} + |\Phi^+\rangle_{A_1a_1}|\Psi^-\rangle_{B_1b_2}\big|\chi^{10}\big\rangle_{a_2b_1} \\ &\quad +|\Phi^+\rangle_{A_1a_1}|\Phi^+\rangle_{B_1b_2}\big|\chi^{11}\big\rangle_{a_2b_1} + |\Phi^+\rangle_{A_1a_1}|\Phi^-\rangle_{B_1b_2}\big|\chi^{12}\big\rangle_{a_2b_1} \\ &\quad +|\Phi^-\rangle_{A_1a_1}|\Psi^+\rangle_{B_1b_2}\big|\chi^{13}\big\rangle_{a_2b_1} + |\Phi^-\rangle_{A_1a_1}|\Psi^-\rangle_{B_1b_2}\big|\chi^{14}\big\rangle_{a_2b_1} \\ &\quad +|\Phi^-\rangle_{A_1a_1}|\Phi^+\rangle_{B_1b_2}\big|\chi^{15}\big\rangle_{a_2b_1} + |\Phi^-\rangle_{A_1a_1}|\Phi^-\rangle_{B_1b_2}\big|\chi^{16}\big\rangle_{a_2b_1}\Big] \end{aligned} \tag{8}$$

Where $|\chi^i\rangle_{a_2\ b_1}(i = 1, 2, \ldots, 16)$ are two-qubit states given by the following equations.

$$\begin{aligned} \big|\chi^1\big\rangle_{a_2b_1} &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \\ \big|\chi^2\big\rangle_{a_2b_1} &= \alpha_0\beta_0|00\rangle - \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle - \alpha_1\beta_1|11\rangle \\ \big|\chi^3\big\rangle_{a_2b_1} &= \alpha_0\beta_1|00\rangle + \alpha_0\beta_0|01\rangle + \alpha_1\beta_1|10\rangle + \alpha_1\beta_0|11\rangle \\ \big|\chi^4\big\rangle_{a_2b_1} &= -\alpha_0\beta_1|00\rangle + \alpha_0\beta_0|01\rangle - \alpha_1\beta_1|10\rangle + \alpha_1\beta_0|11\rangle \\ \big|\chi^5\big\rangle_{a_2b_1} &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle - \alpha_1\beta_0|10\rangle - \alpha_1\beta_1|11\rangle \\ \big|\chi^6\big\rangle_{a_2b_1} &= \alpha_0\beta_0|00\rangle - \alpha_0\beta_1|01\rangle - \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \\ \big|\chi^7\big\rangle_{a_2b_1} &= \alpha_0\beta_1|00\rangle + \alpha_0\beta_0|01\rangle + \alpha_1\beta_1|10\rangle - \alpha_1\beta_0|11\rangle \\ \big|\chi^8\big\rangle_{a_2b_1} &= -\alpha_0\beta_1|00\rangle + \alpha_0\beta_0|01\rangle - \alpha_1\beta_1|10\rangle + \alpha_1\beta_0|11\rangle \\ \big|\chi^9\big\rangle_{a_2b_1} &= \alpha_1\beta_0|00\rangle + \alpha_1\beta_1|01\rangle + \alpha_0\beta_0|10\rangle + \alpha_0\beta_1|11\rangle \\ \big|\chi^{10}\big\rangle_{a_2b_1} &= \alpha_1\beta_0|00\rangle - \alpha_1\beta_1|01\rangle + \alpha_0\beta_0|10\rangle - \alpha_0\beta_1|11\rangle \\ \big|\chi^{11}\big\rangle_{a_2b_1} &= \alpha_1\beta_1|00\rangle + \alpha_1\beta_0|01\rangle + \alpha_0\beta_1|10\rangle + \alpha_0\beta_0|11\rangle \\ \big|\chi^{12}\big\rangle_{a_2b_1} &= -\alpha_1\beta_1|00\rangle + \alpha_1\beta_0|01\rangle - \alpha_0\beta_1|10\rangle + \alpha_0\beta_0|11\rangle \\ \big|\chi^{13}\big\rangle_{a_2b_1} &= -\alpha_1\beta_0|00\rangle - \alpha_1\beta_1|01\rangle + \alpha_0\beta_0|10\rangle + \alpha_0\beta_1|11\rangle \\ \big|\chi^{14}\big\rangle_{a_2b_1} &= -\alpha_1\beta_0|00\rangle + \alpha_1\beta_1|01\rangle + \alpha_0\beta_0|10\rangle - \alpha_0\beta_1|11\rangle \\ \big|\chi^{15}\big\rangle_{a_2b_1} &= -\alpha_1\beta_1|00\rangle - \alpha_1\beta_0|01\rangle + \alpha_0\beta_1|10\rangle + \alpha_0\beta_0|11\rangle \\ \big|\chi^{16}\big\rangle_{a_2b_1} &= \alpha_1\beta_1|00\rangle - \alpha_1\beta_0|01\rangle - \alpha_0\beta_1|10\rangle + \alpha_0\beta_0|11\rangle \end{aligned} \tag{9}$$

Second stage of the protocol is transmission of the transformed single superposition-qubit states using a shared channel between participants. Figure 2 shows the schematic diagram of information transmission in the protocol. As illustrated, Alice and Bob perform Bell-State measurement (BSM) on qubit pairs A_1, a_1 and B_1, b_2, respectively. The remaining state of the system a_2, b_1, as presented in Table 1, will collapse into one of 16 possible states with same probability, depend on Alice and Bob measurement results.

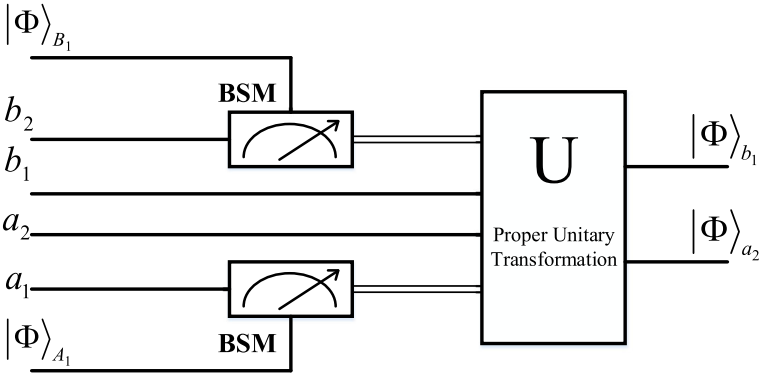

Fig. 2 Schematic diagram for teleportation of single superposition states via four-qubit cluster state and recovering the former by applying appropriate unitary transformations. BSM corresponds to Bell-state measurement, single lines carries quantum bits, double lines represent classical channel which communicates Alice and Bob measurement results

Then Alice and Bob announce their measurement results to each other via a classical channel. Based on the consequence, proper unitary operations are established to reconstruct the transmissible single superposition states. Details of the unitary operations are displayed in Table 2.

Lastly, another transformation circuit is required in order to recover Alice and Bob's arbitrary number of n-qubit states, undo the early transformation and gain *n*-qubit states information once more. Implemented quantum circuit is depicted in Fig. 3, Alice and Bob perform CNOT operations and obtain the two-qubit sates where Φ_{a_2}, Φ_{b_1} serve as controlled qubits and ancillary qubits $|0\rangle_{A_2'}$ and $|0\rangle_{B_2'}$ as target particles. Toffoli gates and another pair of

Table 1 Bell-state measurement results and the consequence collapsed state

Alice's results	Bob's Results	Collapsed state of a_2, b_1
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$\alpha_0\beta_0\|00\rangle+\alpha_0\beta_1\|01\rangle+\alpha_1\beta_0\|10\rangle+\alpha_1\beta_1\|11\rangle$
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$\alpha_0\beta_0\|00\rangle-\alpha_0\beta_1\|01\rangle+\alpha_1\beta_0\|10\rangle-\alpha_1\beta_1\|11\rangle$
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$\alpha_0\beta_1\|00\rangle+\alpha_0\beta_0\|01\rangle+\alpha_1\beta_1\|10\rangle+\alpha_1\beta_0\|11\rangle$
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$-\alpha_0\beta_1\|00\rangle+\alpha_0\beta_0\|01\rangle-\alpha_1\beta_1\|10\rangle+\alpha_1\beta_0\|11\rangle$
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$\alpha_0\beta_0\|00\rangle+\alpha_0\beta_1\|01\rangle-\alpha_1\beta_0\|10\rangle-\alpha_1\beta_1\|11\rangle$
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$\alpha_0\beta_0\|00\rangle-\alpha_0\beta_1\|01\rangle-\alpha_1\beta_0\|10\rangle+\alpha_1\beta_1\|11\rangle$
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$\alpha_0\beta_1\|00\rangle+\alpha_0\beta_0\|01\rangle+\alpha_1\beta_1\|10\rangle-\alpha_1\beta_0\|11\rangle$
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$-\alpha_0\beta_1\|00\rangle+\alpha_0\beta_0\|01\rangle-\alpha_1\beta_1\|10\rangle+\alpha_1\beta_0\|11\rangle$
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$\alpha_1\beta_0\|00\rangle+\alpha_1\beta_1\|01\rangle+\alpha_0\beta_0\|10\rangle+\alpha_0\beta_1\|11\rangle$
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$\alpha_1\beta_0\|00\rangle-\alpha_1\beta_1\|01\rangle+\alpha_0\beta_0\|10\rangle-\alpha_0\beta_1\|11\rangle$
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$\alpha_1\beta_1\|00\rangle+\alpha_1\beta_0\|01\rangle+\alpha_0\beta_1\|10\rangle+\alpha_0\beta_0\|11\rangle$
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$-\alpha_1\beta_1\|00\rangle+\alpha_1\beta_0\|01\rangle-\alpha_0\beta_1\|10\rangle+\alpha_0\beta_0\|11\rangle$
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$-\alpha_1\beta_0\|00\rangle-\alpha_1\beta_1\|01\rangle+\alpha_0\beta_0\|10\rangle+\alpha_0\beta_1\|11\rangle$
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$-\alpha_1\beta_0\|00\rangle+\alpha_1\beta_1\|01\rangle+\alpha_0\beta_0\|10\rangle-\alpha_0\beta_1\|11\rangle$
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$-\alpha_1\beta_1\|00\rangle-\alpha_1\beta_0\|01\rangle+\alpha_0\beta_1\|10\rangle+\alpha_0\beta_0\|11\rangle$
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$\alpha_1\beta_1\|00\rangle-\alpha_1\beta_0\|01\rangle-\alpha_0\beta_1\|10\rangle+\alpha_0\beta_0\|11\rangle$

Table 2 Proper unitary operation for each of the corresponding results

Alice's results	Bob's Results	Collapsed state of a_2, b_1	Alice's operations	Bob's operations
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle+\alpha_1\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle+\beta_1\|1\rangle)_{b_1}$	I	I
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle+\alpha_1\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle-\beta_1\|1\rangle)_{b_1}$	I	Z
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle+\alpha_1\|1\rangle)_{a_2}\otimes(\beta_1\|0\rangle+\beta_0\|1\rangle)_{b_1}$	I	X
$\|\Psi^+\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle+\alpha_1\|1\rangle)_{a_2}\otimes(-\beta_1\|0\rangle+\beta_0\|1\rangle)_{b_1}$	I	iY
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle-\alpha_1\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle+\beta_1\|1\rangle)_{b_1}$	Z	I
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle-\alpha_1\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle-\beta_1\|1\rangle)_{b_1}$	Z	Z
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle-\alpha_1\|1\rangle)_{a_2}\otimes(\beta_1\|0\rangle+\beta_0\|1\rangle)_{b_1}$	Z	X
$\|\Psi^-\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$(\alpha_0\|0\rangle+\alpha_1\|1\rangle)_{a_2}\otimes(-\beta_1\|0\rangle+\beta_0\|1\rangle)_{b_1}$	I	iY
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$(\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle+\beta_1\|1\rangle)_{b_1}$	X	I
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$(\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle-\beta_1\|1\rangle)_{b_1}$	X	Z
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$(\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(\beta_1\|0\rangle-\beta_0\|1\rangle)_{b_1}$	X	$-iY$
$\|\Phi^+\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$(\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(-\beta_1\|0\rangle+\beta_0\|1\rangle)_{b_1}$	X	iY
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Psi^+\rangle_{B_1}\ b_2$	$(-\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle+\beta_1\|1\rangle)_{b_1}$	iY	I
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Psi^-\rangle_{B_1}\ b_2$	$(-\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(\beta_0\|0\rangle-\beta_1\|1\rangle)_{b_1}$	iY	Z
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Phi^+\rangle_{B_1}\ b_2$	$(-\alpha_1\|0\rangle+\alpha_0\|1\rangle)_{a_2}\otimes(\beta_1\|0\rangle+\beta_0\|1\rangle)_{b_1}$	iY	X
$\|\Phi^-\rangle_{A_1}\ a_1$	$\|\Phi^-\rangle_{B_1}\ b_2$	$(\alpha_1\|0\rangle-\alpha_0\|1\rangle)_{a_2}\otimes(\beta_1\|0\rangle-\beta_0\|1\rangle)_{b_1}$	$-iY$	$-iY$

ancillary qubits, will determine three-qubit states for participants. Repeating this procedure, final operation with $(n-1)$ controlled-qubits and an ancillary qubit, will lead into the arbitrary n-qubit states for Alice and Bob respectively and the purpose goal of the protocol is accomplished.

Figure 4 is the complete schematic diagram of the proposed BQT protocol. First, Alice and Bob transformed their n-qubit states of information, then single superposition states of the participants was teleported and lastly, n-qubit states were recover successfully. As stated above and shown in Fig. 1, we can transmit an arbitrary number of n qubits in the same channel shown in Fig. 2. That is, although the number of channel qubits is constant, the complexity of the quantum circuits shown in Figs. 1 and 2 serving as coder and decoder, respectively, is dependent on the number of qubits. As a result, the complexity of the proposed scheme is imposed on quantum circuits not the channel.

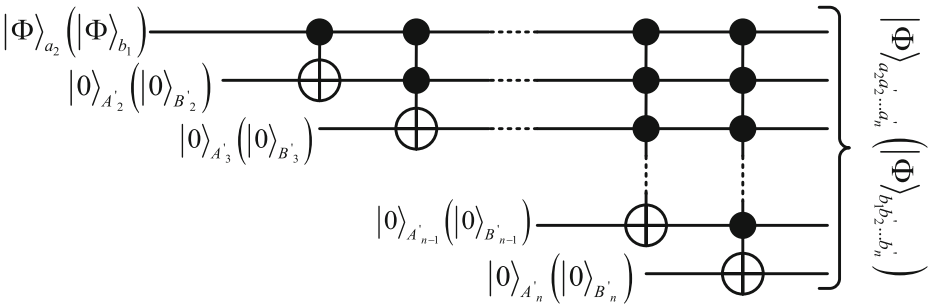

Fig. 3 Quantum circuit for transformation of a single superposition state using auxiliary qubits and controlled-not gates into a class of n-qubit states

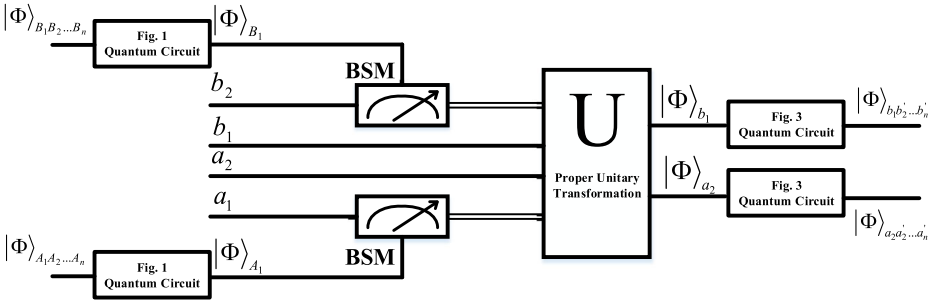

Fig. 4 Schematic diagram for BQT of an arbitrary number of qubits using a four-qubit cluster state

3 Comparison

Efficiency is a significant factor for a proper comparison between proposed protocol and previous works. Eq. 10 defines efficiency [33].

$$\eta = \frac{c}{p+q+t} = \frac{2n}{2n+6} \tag{10}$$

Where c indicates the number of transmitted particles, p is the number of channel qubits, q represents the number of classical bits and t determines the number of auxiliary qubits. In this paper, p and q are constant parameters of the efficiency, while t and c always depend on the number of transmitted qubits between participants. This implies that, as number of transmitted qubits increases efficiency also increase, but result to a higher circuit complexity and noise scale up. Table 3 summarizes and compares the recent BQT and BQCT protocols. Our proposed work compared to the work reported by [34] shows higher efficiency although both works teleport n qubits. Moreover, the work reported by [35] at its best performance shows smaller efficiency compared to our work while that work does not deal with arbitrary number of qubits.

Table 3 Comparison of proposed protocol with previous schemes

Scheme	Number of Teleported qubit states	Quantum channel	$\eta(\%)$
BQCT [9]	1/1	5 qubit composite	28%
BQCT [10]	1/1	6 qubit cluster	25%
BQCT [11]	1/1	7 qubit	22%
BQCT [12]	1/1	6 qubit	25%
BQCT [18]	½	7 qubit	21%
BQCT [20]	2/1	6 qubit cluster	25%
BQT [40]	2/1	5 qubit cluster	33%
BQCT [23]	3/2	9 qubit cluster	27.7%
BQT [35]	2/2	6 qubit	40%
BQT [35]	2/3	6 qubit	45.5%
BQT [36]	1/1	4 qubit cluster	33%
BQCT [37]	3/3	6 qubit GHZ	40%
BQT [41]	2/2	8 qubit	25%
BQT [33]	n/n	2n+2 qubit	40%
BQT [34]	n/n	4n qubits	33%
Proposed Scheme	n/n	4 qubit cluster	66%

4 Conclusion

In this paper, we presented a bidirectional quantum teleportation protocol for simultaneous transmitting an arbitrary number of particles between two legitimate users with exploitation of a four-qubit cluster state as quantum noiseless channel. The main advantages of this protocol are: Implementing simple Bell-state measurements, high efficiency improvement over recent works as discussed in Sect. 3, independency of quantum channel qubits and classical communication resources to the number of transmissible particles with the aid of state-reduction and state-reconstruction circuits. For future work we can add the involvement of a third party called Charlie as supervisor of the protocol, where without Charlie's collaboration Alice and Bob cannot recover the single superposition states. Another future step is the consideration of a noisy channel for bidirectional quantum teleportation. We hope our proposed scheme can be experimentally demonstrated in the future.

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